SOILS AND FOUNDATIONS Vol. 30, No. 1, 65-75, Mar. 1990
Japanese Society of Soil Mechanics and Foundation Engineering

AN ENDOCHRONIC CONSTITUTIVE LAW FOR STATIC SHEAR BEHAVIOR OF OVERCONSOLIDATED CLAYS

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ABSTRACT

Based on the endochronic theory, constitutive relations for overconsolidated clay subjected to static loading have been obtained. The constitutive relations expressed in a closed form can generally describe both deviatoric shear responses and pore pressure responses including negative responses induced by dilatancy. The model further can account for the effect of effective confining pressure on the responses, strain hardening and softening behaviors, dilatancy behaviors, and the effect of overconsolidated ratio, OCR, on the responses. The model is applied to describe the undrained simple shear test data on Boston Blue clay (Ladd and Foott, 1974) and undrained triaxial compression tests on Shanghai clay as well as South Sea clay carried out by the second author. It is shown in this paper that the model yields a reasonable description for the mechanical behaviors of overconsolidated clays subjected to static loading.

Key words : clays, dilatancy, overconsolidation, pore pressure, shear strength, stress-strain curve, triaxial compression test (**IGC** : D 6)

INTRODUCTION

Notable progress of computer analysis has been made in recent years. On the other hand, progress in studying constitutive model of soil is, however, not remarkable. Since a good result of soil analysis can be obtained when both exact numerical calculation and realistic model are used, there is an urgent need looking for a more realistic and mathematically more complicated constitutive relation.

Endochronic theory (Valanis, 1971) is an inelastic constitutive relation. It is based on the irreversible thermodynamics in which an internal variable theory is used as its elementary framework. By considering the thermodynamic restrictive conditions about internal variables, a law about the variations of internal variables can be derived. And since the irreversible change of internal structures of a material can be expressed by internal variables, the constitutive equations are given. The central concept of the endochronic theory is intrinsic time which geometrically represents the length of a state

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Manuscript was received for review on October 11, 1988.
Written discussions on this paper should be submitted before October 1, 1990, to the Japanese Society of Soil Mechanics and Foundation Engineering, Sugayama Bldg. 4 F, Kanda Awaji-cho 2-23, Chiyoda-ku, Tokyo 101, Japan. Upon request the closing date may be extended one month.

path of a material point which traces in the strain-time space. The theory requires no concept of the yield surface as a prior hypothesis, however, it dosen't exclude an apparent yield surface which is obtained as a result derived from the theory. The theory can simulate and predict the most of mechanical behaviors of soils ; especially it can be conveniently and flexibly applied to many loading conditions such as unloading and cyclic loading. Although the theory has some shortcomings, its future and potentiality might be supported by the up to date works of Valanis, Bažant, Wu, and Fan et al..

It is since 1976 that the endochronic theory has been used to model the behaviors of soil. The theory was first applied to describe inelastic behaviors of cohesionles soil by Bažant and Krizek (1976), and later by Ansal and Bažant et al. (1979) to describe visco-plastic behaviors of normally consolidated isotropic clay and of transversely isotropic clay. In this paper, an attempt is made to extend the theory to describe the behaviors of overconsolidated clays.

Up to date various endochronic models for soil have been proposed. The representative models among them may be Bažant's model (1976, 1979) and Wu's model (1983).

Bažant's model (1976, 1979) has been used to describe the behavior of sand and normally consolidated isotropic clay as well as of transversely isotropic clay. The model succeeded in describing such behaviors as strain hardening, stain softening, densification, dilatancy, frictional aspects, and strain-rate dependence of the responses etc.. The model, however, uses many material parameters the number of which reaches 18 for normally consolidated clay, a much complex hardening-softening function and a densification-dilatancy function ; furthermore the model is expressed by a differential type of constitutive equations. These make the determination of parameters difficult and numerical calculation based on the model much complex.

Wu's model (1983), which is expressed by analytical constitutive equations, uses less parameters, the simpler hardening-softening and densification-dilatancy functions. However, the model is constructed for sand only. In addition, the model is based on the Gibbs free energy. Since the Gibbs free energy uses stress components as its variables, it is difficult in principle to describe strain softening behavior of clay subjected to strain-control loading. On the other hand, strain softening behavior may be described by use of the Helmholtz free energy in which strain components are used as its variables (Valanis, 1975).

In this paper, a new model is proposed. Although its general theoretical development follows Wu and Wang (1983), there are the following three basic differences ; (1) the new model is based on the Helmholtz free energy which can reflect strain softening behaviors of clays, (2) normalized equations in which no material parameter depends on effective confining pressure are used, therefore the work to determine the parameters is simplified, (3) negative pore pressure phenomenon can be taken into considerations.

EXPERIMENTAL PROCEDURE

Remoulded Shanghai clay and South Sea clay were used for tests. The Shanghai clay has a plasticity index of about 12% and water content of about 31%. The South Sea clay has a plasticity index of about 8% and water content of about 20%. The Shanghai samples were isotropically preconsolidated under two confining pressures (3 kgf/cm^2 (294.3 kN/m^2) and 5 kgf/cm^2 (490.5 kN/m^2)), and then isotropically unloaded to various stresses in order to produce various OCR-values of 1, 2, 4, and 10 (Table 1.). The South Sea specimens were made by isotropic consolidation under two confining pressures (3 kgf/cm^2 (294.3 kN/m^2) and 4 kgf/cm^2 (392.4 kN/m^2)), followed by isotropically swelling to give three OCR-values of 1, 4, and 10 (Table 2.).

All the shear tests were carried out in

Table 1. stress conditions during consolidation of triaxial tests on Shanghai clay (1 kgf/cm^2=98.1 kN/m^2)

OCR	Preconsolidation	Swelling after preconsolidation
	P_0 (kgf/cm^2)	σ_c' (kgf/cm^2)
1	3	3
	5	5
2	3	1.5
	5	2.5
4	3	0.75
	5	1.25
10	3	0.3
	5	0.5

Table 2. Stress conditions during consolidation of trixaxial tests on South Sea clay (1 kgf/cm^2=98.1 kN/m^2)

OCR	Preconsolidation	Swelling after preconsolidation
	P_0 (kgf/cm^2)	σ_c' (kgf/cm^2)
1	3	3
	4	4
4	3	0.75
	4	1
10	3	0.3
	4	0.4

triaxial apparatus with an axial strain rate of 4.68%/hr under undrained condition.

GENERAL THEORY OF INTERNAL STATE VARIABLES AND FREE ENERGY

Test results which have been reported up to date show that overconsolidated clay has the following general properties :

(1) A tendency of initial strain-hardening followed by strain-softening can be seen in deviatoric response.

(2) Negative proe pressure appears when OCR is large.

(3) Both deviatoric response and volumetric response are influenced by initial effective consolidation stress σ_c'.

(4) For many kinds of clays, the curves of pore pressure-strain and stress-strain can be normalized with respect to σ_c', and a relation between undrained shear strength (or final pore pressure value) and OCR can be found.

Regarding irreversible deformation the following two mechanisms have been recognized :

(1) Inelastic deviatoric response arisen by shear ;

(2) Inelastic volumetric strain caused by shear ;

where volumetric strain caused by volumetric stress is assumed completely reversible, because overconsolidated clay is treated.

According to the internal variable theroy (for example, Valanis, 1971), the macroscopic inelastic behavior of a material can be considered as a result of the some irreversible motion of internal organization and the rearrangement of internal structrue of the material, and the macroscopic inelastic behavior can be described by internal state variables. That is, internal variables $\boldsymbol{p}^s$ (s is the number of the variables) are taken into consideration to reflect the effect of the mechanism (1), and a corresponding intrinsic time Z_s is defined ; further internal variables $q_{kk}{}^d$ (d is the number of the variables) are taken into account to reflect the effect of the mechanism (2), and a corresponding intrinsic time Z_D is defined. Then, when deformation is isothermal, independent internal variables $\boldsymbol{q}^n$ ($n=s+d$) and elementary state variables such as strain tensor $\boldsymbol{\varepsilon}$ together determine uniquely the state of an irreversible system. Therefore, the Holmholtz free energy of a material ψ can be expressed as a function of $\boldsymbol{q}^n$ and $\boldsymbol{\varepsilon}$ as follows :

$$\psi=\psi(\boldsymbol{\varepsilon}, \boldsymbol{q}^n) \quad (1)$$

When considering initially isotropic clay subjected to small strain, the free energy ψ can be separated into two parts, i. e.:

$$\psi(\boldsymbol{\varepsilon}, \boldsymbol{q}^n)=\psi_H(\varepsilon, q^d)+\psi_D(\boldsymbol{\varepsilon}^D, \boldsymbol{q}^s) \quad (2)$$

where, the subscripts H and D denote, respectively, the hydrostatic and deviatoric parts ; $\varepsilon=\varepsilon_{kk}/3$; $q^d=q_{kk}{}^d/3$; and $\boldsymbol{\varepsilon}^D$ is deviatoric strain tensor.

The material parameters in ψ_H and ψ_D are dependent on effective consolidation stress. To facilitate the procedure determining the parameters the normalized property of clay is used, and then Eq. (2) is rewritten by the nondimensional form :

$$\bar{\psi}(\varepsilon, \boldsymbol{q}^n)=\bar{\psi}_H(\varepsilon, q^d)+\bar{\psi}_D(\boldsymbol{\varepsilon}^D, \boldsymbol{p}^s) \quad (3)$$

In this nomalized equation, the material parameters are no longer dependent on σ_c'.

The function $\bar{\psi}$ acts as a potential function of the irreversible system, and normalized deviatoric stress, $\bar{\boldsymbol{s}}$, and effective hydrostatic stress, $\bar{\sigma}_{kk}'$, can be given by

$$\left.\begin{aligned} \bar{\boldsymbol{s}}&=\boldsymbol{s}/\sigma_c'=\partial\bar{\psi}_D/\partial\boldsymbol{\varepsilon}^D \\ \bar{\sigma}_{kk}'&=\sigma_{kk}'/\sigma_c'=\partial\bar{\psi}_H/\partial\varepsilon \end{aligned}\right\} \quad (4)$$

When expanding $\bar{\psi}_H$ and $\bar{\psi}_D$ to two orders Taylor series, the following equations can be obtained :

$$\left.\begin{aligned} \bar{\psi}_H&=A\varepsilon^2/2-\sum_d B^d q^d\varepsilon+\sum_d C^d q^d q^d/2 \\ \bar{\psi}_D&=\boldsymbol{\varepsilon}^D \underset{\approx}{D} \boldsymbol{\varepsilon}^D/2-\sum_s \boldsymbol{\varepsilon}^D \underset{\approx}{E}^s \boldsymbol{p}^s+\sum_s \boldsymbol{p}^s \underset{\approx}{F}^s \boldsymbol{p}^s/2 \end{aligned}\right\} \quad (5)$$

where A, B^d and C^d are positive material constants ; $\underset{\approx}{D}, \underset{\approx}{E}^s$ and $\underset{\approx}{F}^s$ are fourth-orders material constant tensors ; $\underset{\approx}{D}$ is semi-positive definite ; and $\underset{\approx}{F}^s$ is positive definite.

Substituting Eq. (5) into Eq. (4) and assuming $\underset{\approx}{D}, \underset{\approx}{E}^s$, and $\underset{\approx}{F}^s$ be isotropy matrix (i. e., $M_{ijkl}=M_1\delta_{ij}\delta_{kl}+M_2\delta_{ik}\delta_{jl}$), the stresses can be expressed as functions about $\boldsymbol{\varepsilon}$ and $\boldsymbol{q}^n$ by

$$\left.\begin{aligned} \bar{\boldsymbol{s}}&=D_2\boldsymbol{\varepsilon}^D-\sum_s E_2^s\boldsymbol{p}^s \\ \bar{\sigma}_{kk}'&=A\varepsilon-\sum_d B^d q^d \end{aligned}\right\} \quad (6)$$

According to the Clausius-Duhem inequality, the variations in internal variables must meet the restriction :

$$\sum_{i=1}^{n}\left[-\frac{\partial\bar{\psi}}{\partial\boldsymbol{q}^i}d\boldsymbol{q}^i\right]\geq 0 \quad (7)$$

Because the n number of internal variables $\boldsymbol{q}^n$ are independent with each other, the restriction (7) becomes

$$-\frac{\partial\bar{\psi}}{\partial\boldsymbol{q}^i}\frac{d\boldsymbol{q}^i}{dt}\geq 0 \quad (i=1, 2, \cdots, n) \quad (8)$$

where t is time. Due to $dZ/dt>0$, these inequalities can be rewritten as

$$-\frac{\partial\bar{\psi}}{\partial\boldsymbol{q}^i}\frac{d\boldsymbol{q}^i}{dz}\geq 0 \quad (i=1, 2, \cdots, n) \quad (9)$$

Under the condition of small deformation, it can be verified that the inequalities (9) is satisfied if the following relations exist :

$$\frac{\partial\bar{\psi}}{\partial\boldsymbol{q}^i}+\boldsymbol{b}^i\frac{d\boldsymbol{q}^i}{dz}=0 \quad (i=1, 2, \cdots, n) \quad (10)$$

in which $\boldsymbol{b}^i$ is positive constant tensor. Eq. (10) can also be written as

$$\left.\begin{aligned} \frac{\partial\bar{\psi}_D}{\partial\boldsymbol{p}^i}+a^i\frac{d\boldsymbol{p}^i}{dz_s}&=0 \quad (i=1, 2, \cdots, s) \\ \frac{\partial\bar{\psi}_H}{\partial q^i}+b\frac{dq^i}{dz_D}&=0 \quad (i=1, 2, \cdots, d) \end{aligned}\right\} \quad (11)$$

where a^i and b^i are positive constants.

Combining Eqs. (5), (6), and (11), the stresses $\bar{\boldsymbol{s}}$ and $\bar{\sigma}_{kk}'$ can be expressed as the function about total strain $\boldsymbol{\varepsilon}$ and intrinsic time Z by

$$\bar{\boldsymbol{s}}=D_2\boldsymbol{\varepsilon}^D-\sum\mu_s\int_0^{Z_s}\exp[-\rho^s(Z_s-Z_s')] \times\boldsymbol{\varepsilon}^D(Z_s')dZ_s' \quad (12)$$

and

$$\bar{\sigma}_{kk}'=A\varepsilon-\sum_d k^d\left\{\varepsilon-\varepsilon(0)\exp(-\lambda^d Z_D) -\int_0^{Z_D}\exp[-\lambda^d(Z_D-Z_D')]\frac{d\varepsilon(Z_D')}{dZ_D'}dZ_D'\right\} \quad (13)$$

where

$\mu_s=(E_2^s)^2/a^s\geq 0$; $\rho^s=F_2^s/a^s>0$;
$k^d=(B^d)^2/c^d\geq 0$; and $\lambda^d=c^d/b^d\geq 0$.

DEVIATORIC RESPONSE

General Equation of Deviatoric Response

By use of the Laplace transformation, Eq. (12) is expressed in terms of plastic strain $\boldsymbol{\varepsilon}^p$ as follows :

$$\bar{\boldsymbol{s}}=2G\rho_0\frac{d\boldsymbol{\varepsilon}^p}{dZ_s}+\int_0^{Z_s}\rho_1(Z_s-Z_s')\frac{d\boldsymbol{\varepsilon}^p}{dZ_s'}dZ_s'+\mu_2\boldsymbol{\varepsilon}^p \quad (14)$$

in which $\rho_1(Z)=\sum_{r=1}^{s-1}R_r\exp(-\beta_r Z)$; $\boldsymbol{\varepsilon}^p=\boldsymbol{\varepsilon}^D-\bar{\boldsymbol{s}}/2G$; ρ_0, R_r, β_r, and μ_2 are positive material constants ; and D_2 has been set equal to shear modulus $2G$ taking initial elastic behavior into account.

To reflect the strain hardening-softening phenomena in deviatoric response on the model, another intrinsic time scale is introduced in the form

$$dZ_s = F(\zeta_s)\,d\zeta_s \qquad (15)$$

in which $F(\zeta_s)$ is the hardening-softening function ; and ζ_s, which serves to record the deformation history, is governed by

$$d\zeta_s^2 = d\varepsilon^p \cdot d\varepsilon^p \qquad (16)$$

Restricting attention to one dimensional monotonic loading and employing only one term of the function $\rho_1(Z)$, Eq. (14) can be written as

$$\bar{s} = 2G\rho_0 \frac{d\zeta_s}{dZ_s} + \int_0^{Z_s} \mu_1 \exp[-\alpha(Z_s - Z_s')] \times \frac{d\zeta_s'}{dZ_s'} dZ_s' + \mu_2 \varepsilon^p \qquad (17)$$

and Eq. (16) becomes

$$d\zeta_s^2 = d\varepsilon^p \cdot d\varepsilon^p \qquad (18)$$

where μ_1 and α are positive material constants.

Strain Hardening-softening Function Presently Proposed

$d\zeta_s/dZ_s$ in Eq. (17) is connected with the strain hardening-softening function $F(\zeta_s)$, and $F(\zeta_s)$ has a great effect on predicted curve's shape and final expression of Eq. (17). Therefore, it is important to constitute $F(\zeta_s)$ in successful manner.

For normally consolidated clay, Bažant et al. (1979) suggested a strain hardening-softening function as follows :

$$dZ_s = \frac{d\zeta_s}{Z_1} = \frac{F(\varepsilon, \sigma, \zeta_s)}{Z_1} d\xi = \frac{d\eta}{Z_1 f(\eta)} ;$$
$$d\eta = F_\eta(\varepsilon, \sigma)\,d\xi \qquad (19)$$

in which Z_1=constant ; σ=stress tensor ; $d\xi = \sqrt{J_2^{d\varepsilon}}$;

$$\left.\begin{aligned} f(\eta) &= 1 + \frac{\beta_1 \eta}{1 + \beta_2 \eta} \\ F_\eta(\varepsilon, \sigma) &= a + \frac{|1 - a_1 I_1^\varepsilon|(1 + a_3 J_2^\varepsilon)}{0.01 + a_2 I_1^{\sigma'}/P_a} \end{aligned}\right\} \qquad (20)$$

where $\beta_1, \beta_2, a, a_1, a_2$, and a_3 are positive material constants ; I_1^ε and $I_1^{\sigma'}$ are, respectively, the first invariant of strain and of effective stress ; J_2^ε and $J_2^{d\varepsilon}$ are, respectively, the second invariant of deviatoric strain and of incremental one ; and P_a is atmospheric pressure.

By use of Eqs. (19) and (20), the hardening-softening property of a material can be described, however no analytical expression of Eq. (17) can be obtained ; therefore it is hard to estimate so many initial values of the parameters.

Wu et al. (1980) used the following hardening-softening function for studing the dynamic response of low-carbon steel :

$$\frac{d\zeta_s}{dZ_s} = \begin{cases} \exp(-\beta_s Z_s) & Z_s \le Z_{scr} \\ b_h \exp[\beta_h (Z_s - Z_{scr})] & Z_s \ge Z_{scr} \end{cases} \qquad (21)$$

in which $b_h = 1 - \beta_s \zeta_{scr}$; β_s and β_h are positive material constants ; and Z_{scr} is an intrinsic time point which divides softening response and hardening one.

By use of Eq. (21) an analytical expression of Eq. (17) can be obtained, therefore most of the material constants can be determined easily by using some particular values of a response curve such as the peak stress, the slope of the asymptotic straight line at large strain, and etc.. But Eq. (21) is given stage by stage and when substituting it into Eq. (17) it can be seen that the first order derivative of $s \sim \varepsilon^p$ response curve is not continuous at $\varepsilon^p = \varepsilon^p_{scr}$ (Fig. 1). Therefore, Eq. (21) is not a suitable hardening-softening

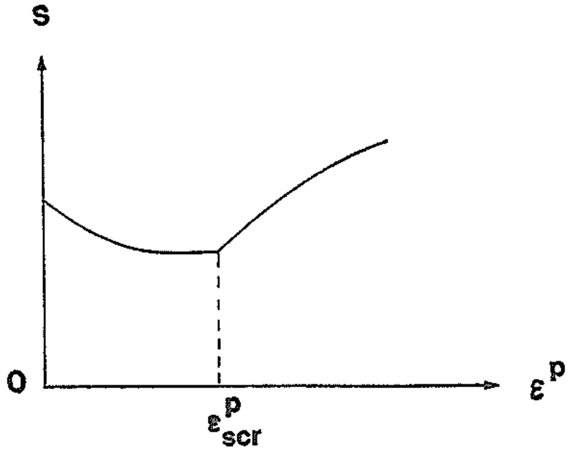

Fig. 1. $s-\varepsilon^p$ **curve at a constant strain rate for b.c.c. metal**

function for a soil showing a smooth response curve.

In this paper, an attempt is made to constitute a hardening-softening function which meets the following requirements.

(1) The characteristics of first hardening and then softening of material can be reflected.

(2) By use of the hardering-softening function an analytical expression of the deviatoric constitutive Eq. (12) can be obtained, thus the parameters can be determined easily.

(3) For the purpose to guarantee that predicted response curves are smooth and Eq. (12) can be used conveniently in more complex stress calculation, the function must be expressed in a coninuous form and not in a stage by stage form.

To satisfy the requirement (1), it is good to separate the hardening-softening function into a hardening function $f_h(Z_s)$ and a softening one $f_s(Z_s)$ (Valanis, 1974) as follows :

$$\frac{d\zeta_s}{dZ_s}=f_h(Z_s)\cdot f_s(Z_s) \qquad (22)$$

Regarding the hardening function it may be the simplest possible choice to assume the linear relationship $f_h(Z_s)=(1+\alpha_s Z_s)$. Regarding the softening function, on the other hand, the following forms have been proposed : $(1-\beta Z_s)$, $1/(1+\beta Z_s)$ (Valanis, 1974) and $\exp(-\beta Z_s)$ (Wu, 1980). In this paper the last one is adopted because it meets both the requirement (1) and (2). Therefore, the hardening-softening function presently proposed becomes

$$d\zeta_s/dZ_s=(1+\alpha_s Z_s)\exp(-\beta_s Z_s); \quad (\alpha_s, \beta_s>0) \qquad (23)$$

Differentiating Eq. (23) with respect to Z_s, the following equation is obtained :

$$d^2\zeta_s/dZ_s^2=(\alpha_s-\beta_s-\beta_s\alpha_s Z_s)\exp(-\beta_s Z_s)$$

$d^2\zeta_s/dZ_s^2=0$ yields

$$Z_s^*=(\alpha_s-\beta_s)/(\beta_s\alpha_s) \qquad (24)$$

When $Z_s\leq Z_s^*$, $d^2\zeta_s/dZ_s^2\geq 0$, which corresponds to hardening behavior ; when $Z_s>Z_s^*$, $d^2\zeta_s/dZ_s^2<0$, softening behavior begins and

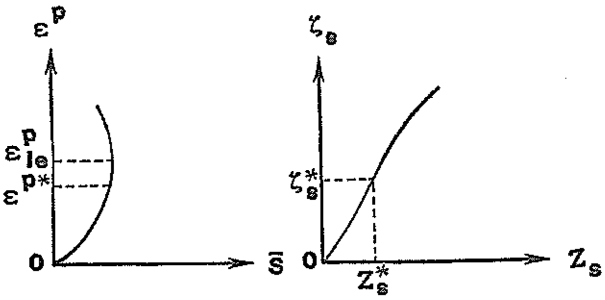

Fig. 2. The curve of ζ_s-Z_s and $\varepsilon^p-\bar{s}$

the first order derivation of $\bar{s}-\varepsilon^p$ curve gradually varies from positive to negative as shown in Fig. 2.

Although ε^{p*} is not equal to ε_{le}^p, which corresponds to $d\bar{s}/d\varepsilon^p=0$, it is not necessary to determine ε^{p*} and it is required only to select α_s and β_s to meet

$$(\alpha_s-\beta_s)(\beta_s\alpha_s)<\varepsilon_{le}^p \qquad (25)$$

The above analysis has showed that Eq. (23) can describe strain hardening-softening properties of soil. When Eqs. (23) and (18) are substituted into Eq. (17), the following relations are obtained :

$$\begin{aligned}\bar{s}&=2G\rho_0(1+\alpha_s Z_s)\exp(-\beta_s Z_s)+\mu_1\exp(-\alpha Z_s)\\&\quad\times\int_0^{Z_s}(1+\alpha_s Z_s')\exp[(\alpha-\beta_s)Z_s']dZ_s'+\mu_2\zeta_s\\&=2G\rho_0(1+\alpha_s Z_s)\exp(-\beta_s Z_s)\\&\quad+\frac{\mu_1}{\alpha-\beta_s}\exp(-\alpha_s Z_s)\int_0^{Z_s}(1+\alpha_s Z_s')\\&\quad\cdot d\{\exp[(\alpha-\beta_s)Z_s']\}+\mu_s\zeta_s\end{aligned} \qquad (26)$$

By integration by parts, this equation is further written as

$$\begin{aligned}\bar{s}&=2G\rho_0(1+\alpha_s Z_s)\exp(-\beta_s Z_s)\\&\quad+\frac{\mu_1}{\alpha-\beta_s}\exp(-\alpha Z_s)\Big[(1+\alpha_s Z_s)Y-1\\&\quad-\frac{\alpha_s(Y-1)}{\alpha-\beta_s}\Big]+\mu_2\zeta_s\end{aligned} \qquad (27)$$

where $Y=\exp[(\alpha-\beta_s)Z_s]$. This equation is a single analytical constitutive equation.

Determination of Material Parameters

In Eq. (25), ρ_0, α_s, and β_s are constant parameters to be determined, and $2G$, μ_1, μ_2, and α depend on OCR.

According to the data published in the literatures (Ladd and Foott, 1974), (Henkel, 1956), and (Parry, 1960), a correlation exists between the undrained shear strength of clay $\bar{s}_u$ and OCR. It can be expressed by the following exponential function.

$$\bar{s}_u=(\bar{s}_u)_{NC}\cdot \mathrm{OCR}^{m_1} \qquad (28)$$

in which NC denotes normally consolidation, m_1 is a material constant determined by test.

Furthermore, to simplify the procedures of determining the parameters, it is assumed that there exist the following approximated correlations for $\varepsilon_{le}{}^p$, $2G^p$, and $2G$.

$$\left.\begin{aligned} \varepsilon_{le}{}^p&=f_1(\mathrm{OCR})\\ 2G^p&=f_2(\mathrm{OCR})\\ 2G&=f_3(\mathrm{OCR})\end{aligned}\right\} \qquad (29)$$

in which ε_{lep} is the value of ε^p corresponding to $d\bar{s}/d\varepsilon^p=0$; $2G^p=(d\bar{s}/d\varepsilon^p)_{\varepsilon^p=0}$; and functions f_1, f_2, and f_3 are determined by experimental data. It should be here mentioned that $\varepsilon_{le}{}^p$, $2G^p$ and $2G$ are defined for a normalized curve; therefore, the functions f_1, f_2, and f_3 are independent of σ_c'.

Then μ_1 μ_2, and α can be connected with $2G^p$, $\bar{s}_u$, and $\varepsilon_{le}{}^p$ by the following equations derived from Eq. (27):

$$\left.\begin{aligned} &2G\rho_0(\alpha_s-\beta_s)+\mu_1+\mu_2=2G^p\\ &\left\{2G\rho_0(\alpha_s-\beta_s-\beta_s\alpha_s Z_s)-\frac{\alpha\mu_1}{(\alpha-\beta_s)Y}\right.\\ &\quad\times\left[(1+\alpha_s Z_s)Y-1-\frac{\alpha_s}{\alpha-\beta_s}(Y-1)\right]\\ &\quad\left.+(\mu_1+\mu_2)(1+\alpha_s Z_s)\right\}_{Z_s=Z_{le}}=0\\ &\left\{2G\rho_0(1+\alpha_s Z_s)\exp(-\beta_s Z_s)\right.\\ &\quad+\frac{\mu_1}{\alpha-\beta_s}\exp(-\alpha Z_s)\left[(1+\alpha_s Z_s)Y-1\right.\\ &\quad\left.\left.-\frac{\alpha_s}{\alpha-\beta_s}(Y-1)\right]+\mu_2\zeta_s\right\}_{Z_s=Z_{le}}=\bar{s}_u \end{aligned}\right\} \qquad (30)$$

where Z_{le} is the value of Z_s corresponding to $\varepsilon_{le}{}^p$.

The material constant ρ_0 is determined by the yield stress in the case of OCR$=1$($\sigma_y=2G\rho_0$), and α_s and β_s are determined by the trial and error method so that Eq. (27) well fits experimental data.

The full lines in Fig. 3 show theoretical results obtained from Eq. (27), and corresponding experimental data are the one obtained from undrained simple shear tests after K_0 consolidation on Boston Blue clay (Ladd and Foott, 1974). It is seen that a good agreement has been achieved. The constants are determined to be $\rho_0=0.0078$; $\alpha_s=10$; and $\beta_s=7$ by fitting the OCR$=1$ curve. Remaining the constants unchanged, Eq. (27) predicted the other curves.

The material functions m_1, f_1, and f_2 are found to be

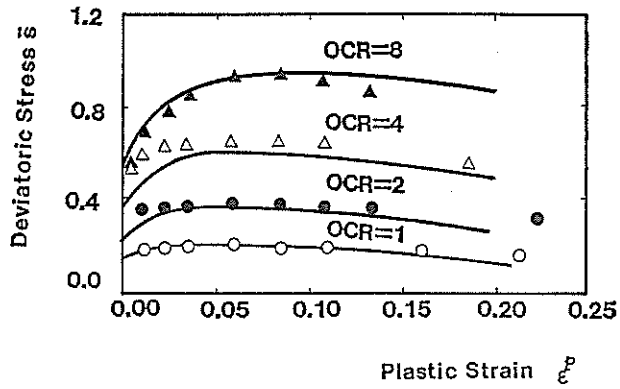

Fig. 3. Deviatoric response data for Boston Blue clay (after Ladd and Foott, 1974) with theoretical response

Table 3. Material parameters of deviatoric response for Boston Blue clay

OCR	ρ_0	α_s	β_s	α	μ_1	μ_2
1	0.0078	10	7	263.7	19.6	0.046
2	0.0078	10	7	158.0	20.4	0.163
4	0.0078	10	7	107.7	22.1	0.513
8	0.0078	10	7	95.2	25.4	1.447

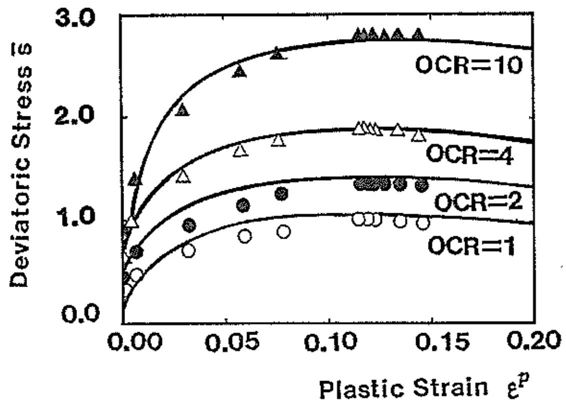

Fig. 4. Deviatoric response data for Shanghai clay with theoretical response

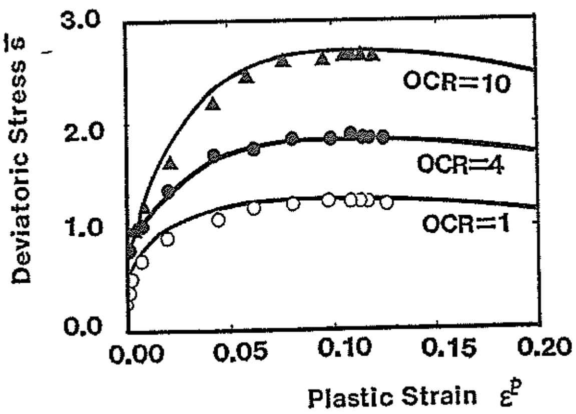

Fig. 5. Deviatoric response data for South Sea clay with theoretical response

$$m_1=0.1\exp[-0.01(\mathrm{OCR}^3-1)]+0.75$$

$$f_1=0.045+0.0069\cdot\mathrm{OCR}$$

$$f_2=18.8+1.2\cdot\mathrm{OCR}$$

based on the data. Where as an approximation, $2G$ is expressed by the same function as $\bar{s}_u$. The material constants obtained and used are shown in Table 3.

The full lines in Figs. 4 and 5 are theoretical results obtained from Eq. (27). The experimental data were obtained, respectively, for Shanghai clay and South Sea clay by carrying out undrained triaxial compression tests. The material functions are found to be for Shanghai clay :

$$m_1=0.43$$

$$f_1=0.1170+0.0019\cdot\mathrm{OCR}$$

$$f_2=46+91.1\cdot\mathrm{OCR}$$

for South Sea clay :

Table 4. Material parameters of deviatoric response for Shanghai clay

OCR	ρ_0	α_s	β_s	α	μ_1	μ_2
1	0.0043	10.3	6.9	67.63	43.63	1.77
2	0.0043	10.3	6.9	89.37	76.65	2.52
4	0.0043	10.3	6.9	104.11	118.70	3.54
10	0.0043	10.3	6.9	75.14	122.80	5.57

Table 5. Material parameters of deviatoric response for South Sea clay

OCR	ρ_0	α_s	β_s	α	μ_1	μ_2
1	0.0063	10.4	4.2	109.36	93.48	1.92
4	0.0063	10.4	4.2	96.18	121.20	2.98
10	0.0063	10.4	4.2	82.06	143.17	4.59

$$m_1=0.28865+0.00625(\mathrm{OCR}-4)$$

$$f_1=0.1023+0.00152\cdot\mathrm{OCR}$$

$$f_2=96\cdot\mathrm{OCR}^{0.191}$$

The material constants obtained and used are listed in Tables 4. and 5..

PORE PRESSURE RESPONSE

The variations of pore pressure along a strain path can be formulated from the volumetric response expressed by Eq. (13).

It has been known that densification-dilatancy grows even when unloading takes place in shear. Therefore, it is suitable to define the intrinsic time, dZ_D, as follows :

$$dZ_D=d\zeta_D/(1+\xi\zeta_D)\ ;\ (\xi>0) \quad (31)$$

$$d\zeta_D{}^2=d\varepsilon^D\cdot d\varepsilon^D \quad (32)$$

In the special case of constant confined-pressure triaxial test, Eq. (32) becomes

$$d\zeta_D{}^2=d\varepsilon_a\cdot d\varepsilon_a \quad (33)$$

where ε_a is accumulated total axial strain after the start of shear.

When restricting attention to undrained shear test with $\varepsilon=\varepsilon(0)=$Const., and specifying that $d=1$ for simplicity, Eq. (13) becomes

$$\bar{\sigma}_{kk}'=A\varepsilon(0)-k[\varepsilon(0)-\varepsilon(0)\exp(-\lambda Z_D)] \quad (34)$$

When $Z_D=0$, $\bar{\sigma}_{kk}'=2(\sigma_r'+\sigma_a')/\sigma_c'=3\sigma_c'/\sigma_c'=3$ and Eq. (34) yields

$$\varepsilon(0)=3/A \quad (35)$$

When combining Eq. (34) and (35), the following equation is obtained :

$$\bar{\sigma}'=1-c[1-\exp(-\lambda Z_D)] \quad (36)$$

where $c=k/A>0$; and $\bar{\sigma}'=\bar{\sigma}_{kk}'/3$.

By applying the effective stress principle and substituting Eq. (31) and (33) for Eq. (36), the pore pressure can be expressed as

$$\bar{u}=u/\sigma_c'=\bar{\sigma}-1+c[1-(1+\xi\varepsilon_a)^{-\lambda/\xi}] \quad (37)$$

in which u is pore pressure.

Eq. (37) is an analytical formula which expresses pore pressure in terms of applied total stress $\bar{\sigma}$ and the accumulation of toal axial strain ε_a. But it is found that the formula is not enough to describe complex pore pressure responses of overconsolidated clay ; especially, a negative pore pressure behavior occuring in a constant confined-pressure triaxial test on high OCR clay. In such a test of constant confined-pressure (i.e., $\sigma_r=\sigma_c'$)

$$\sigma_a>\sigma_r$$

$$\bar{\sigma}=(\sigma_a+2\,\sigma_r)/\sigma_c'=(\sigma_a+2\,\sigma_r)/3\,\sigma_r>1$$

and Eq. (37) yields $\bar{u}$ which is always positive.

In order to effectively describe the pore pressure behavior of overconsolidated clay the method proposed by Valanis and Wu (1975) was used ; their study was about the accumulation of axial strain of metal under zero axial stress. In the method it is considered that for the most work hardening material it is allowed to retain the linear terms in the expansion of free energy because the tensile response is different from compression one. By use of the method, the volumetric relation may be rewritten as

$$\bar{\sigma}_{kk}'=A\varepsilon-\sum_d k^d\Big\{\varepsilon-\varepsilon(0)\exp(-\lambda^d Z_D) -\int_0^{Z_D}\exp[-\lambda^d(Z_D-Z_D')]\frac{d\varepsilon(Z_D')}{dZ_D'}dZ_D'\Big\} +\sum_d r^d[1-\exp(-\rho^d Z_D)] \qquad (38)$$

in which r^d, $\rho^d\geq 0$, and the equation is derived in a method similar to Eq. (13). Accordingly, the pore pressure can be expressed as

$$\bar{u}=\bar{\sigma}-1-R[1-(1+\xi\varepsilon_a)^{-\rho/\xi}] +c[1-(1+\xi\varepsilon_a)^{-\lambda/\xi}] \qquad (39)$$

in which $R=\gamma/3>0$. For constant confined-pressure triaxial test, $\sigma_r=\sigma_c'$ thus $\bar{\sigma}=1+\bar{s}/2$ and Eq. (39) becomes

$$\bar{u}=\bar{s}/2-R[1-(1+\xi\varepsilon_a)^{-\rho/\xi}] +c[1-(1+\xi\varepsilon_a)^{-\lambda/\xi}] \qquad (40)$$

Compared with Eq. (37), Eq. (40) has an additional term $-R[1-(1+\xi\varepsilon_a)^{-\rho/\xi}]$. It can be seen in the following analysis that this equation adequately describes the pore pressure responses for different OCR values. Especially, if $\bar{u}-\bar{s}/2$ is found to be a monotonic function of ε_a, Eq. (40) can be simplified as follows by assuming $\rho=\lambda$

$$\bar{u}=\bar{s}/2-Q[(1+\xi\varepsilon_a)^{-\lambda/\xi}-1] \qquad (41)$$

where $Q=c-R$ and R is an arbitrary constant.

The material constants in Eq. (40) may be determined in the manner similar to the above section, that is, ξ and ρ remain constant for different OCR and are determined by trial and error method so that Eq. (40) fits the experimental data of OCR$=$1 ; and the values R, c, and λ are connected indirectly with OCR by the following equations derived from Eq. (40)

$$\left.\begin{aligned} &D_0=G-R\rho+c\lambda\\ &\bar{u}_f=\bar{s}_u/2+R[(1+\xi\varepsilon_{le})^{-\rho/\xi}-1] +c[1-(1+\xi\varepsilon_{le})^{-\lambda/\xi}]\\ &D_f=\left(\frac{d\bar{s}}{d\varepsilon_a}\right)_{\varepsilon_{le}}\Big/2-R\rho(1+\xi\varepsilon_{le})^{-\rho/\xi-1} +c\lambda(1+\xi\varepsilon_{le})^{-\lambda/\xi-1} \end{aligned}\right\} \qquad (42)$$

where

$$\left.\begin{aligned} &D_0=\left(\frac{d\bar{u}}{d\varepsilon_a}\right)_{\varepsilon_a=0}=f_4(\mathrm{OCR})\\ &\bar{u}_f=(\bar{u})_{\varepsilon_{le}}=(\bar{u}_f)_{NC}\cdot(2-\mathrm{OCR}^{m_2})\\ &D_f=\left(\frac{d\bar{u}}{d\varepsilon_a}\right)_{\varepsilon_{le}}=f_5(\mathrm{OCR}) \end{aligned}\right\} \qquad (43)$$

in which m_2, f_4, and f_5 are determined by experimental data ; and ε_{le} is the value of ε_a corresponding to $\varepsilon_{le}{}^p$.

The material parameters in Eq. (41) can be determined by the following relations derived from Eq. (41)

$$\left.\begin{aligned} &D_0=G+Q\lambda\\ &\bar{u}_f=\bar{s}_u/2-Q[(1+\xi\varepsilon_{le})^{-\lambda/\xi}-1]\\ &D_f=\left(\frac{d\bar{s}}{d\varepsilon_a}\right)_{\varepsilon_{le}}\Big/2+Q\lambda(1+\xi\varepsilon_{le})^{-\lambda/\xi-1} \end{aligned}\right\} \qquad (44)$$

The full lines in Figs. 6 and 7 show the theoretical results obtained from Eq. (40) or (41), and the experimental data were obtained from Shanghai clay and South Sea clay.

The material functions are found to be for Shanghai clay :

$$m_2=-0.001803\cdot\mathrm{OCR}+0.57239$$

$$f_4=(2.8\cdot\mathrm{OCR}^2-27.8\cdot\mathrm{OCR}+79)/9$$

$$f_5=-0.7\cdot\mathrm{OCR}+2$$

for South Sea clay :

$$m_2=0.72616-0.0116\cdot\mathrm{OCR}$$

$$f_4=0.1593\cdot\mathrm{OCR}^2-1.863\cdot\mathrm{OCR}+8.7037$$

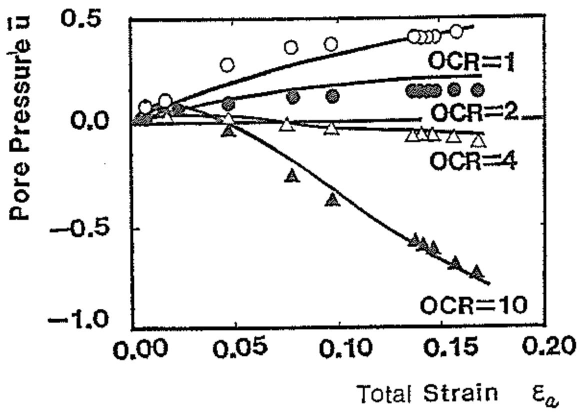

Fig. 6. Pore presure data for Shanghai clay with theoretical response

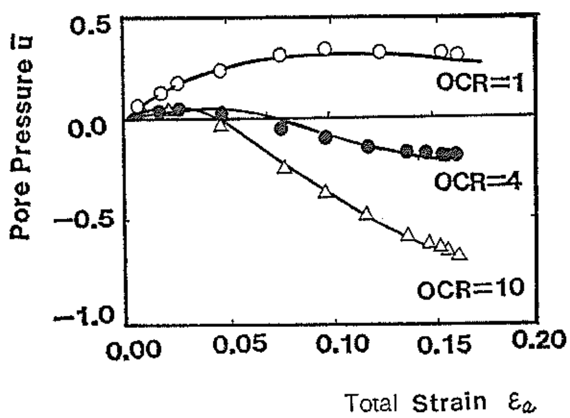

Fig. 7. Pore pressure data for South Sea clay with theoretical response

Table 6. Material parameters of pore pressure response for Shanghai clay

OCR	ξ	ρ	R	c	λ
1	20.0	35.	0.996	3.5	3.58
2	20.0	35.	0.997	1.38	5.21
	ξ	λ	Q		
4	9.46	33.26	−1.67		
10	38.86	1.89	−22.80		

Table 7. Material parameters of pore pressure response for South Sea clay

OCR	ξ	λ	Q
1	1.07	28.11	−32.02
4	2.06	15.58	−1.29
10	1.64	11.01	−2.54

$$f_5=0.3022-0.5022\cdot \mathrm{OCR}$$

The material parameters are presented in Tables 6. and 7..

CONCLUSIONS

Based on the fundamental laws of continuum mechanics, a set of analytical endochronic constitutive equations for overconsolidated clay subjecting to static lading has been derived.

In the developed model, a strain hardening-softening function is proposed. The function not only validly predicts the first hardening followed by softening property in deviatoric response of clay, but also simplifies the procedure to determine parameters because a deviatoric response equation in a single and closed form is used.

The normalization property of clay is used to express the equations in a nondimensional form, which saves us the trouble to find the relations between σ_c' and parameters.

In this model, a new equation is proposed to successfully evaluate negative pore pressure phenomenon by use of a method in which linear terms in the series expansion of free energy are retained.

A total of 12 parameters, including the initial shear modulus, appears in the constitutive equation. However, it is only five of them that must be determined by fitting the test data of OCR=1, and the other seven parameters can be evaluated by the correlations between OCR and several particular values of response such as shear strength, the peak value of pore pressure and etc..

It has been finally shown that the proposed equations are in reasonable agreement with the experimental results of triaxial tests on Shanghai clay and South Sea clay and another test results reported in the reference.

ACKNOWLEDGMENTS

This work is a part of the second author's doctoral work carried out at the University of Tongji, China and the University of Yokohama, Japan. The second author gratefully acknowledges the encouragement and useful comments made by Hou X. Y. and Wei D. D. of Tongji University, China, during the course of her doctoral work.

NOTATIONS

$\boldsymbol{\varepsilon}^D, \boldsymbol{\varepsilon}^p$=total and plastic deviatoric strain
$\varepsilon^D, \varepsilon^p$=total and plastic deviatoric strain for one dimensional case
$\varepsilon_{kk}, \varepsilon$=volumetric and mean strain
ε_a=total axial strain during shearing
$\boldsymbol{s}, \bar{\boldsymbol{s}}$=deviatoric stress ($\bar{\boldsymbol{s}}=\boldsymbol{s}/\sigma_c'$)
$s, \bar{s}$=deviatoric stress for one dimensional case
$\sigma_{kk}, \bar{\sigma}_{kk}, \bar{\sigma}_{kk}', \sigma, \bar{\sigma}, \bar{\sigma}'$=volumetric and mean stress
σ_y=yield stress
σ_c'=effective consolidation stress
$I_1^\varepsilon, I_1, \sigma'$=first invariant of strain and of effective stress
$J_2^\varepsilon, J_2^{d\varepsilon}$=second invariant of deviatoric strain and of incremental one
$\psi, \bar{\psi}, \psi_H, \bar{\psi}_H, \psi_D, \bar{\psi}_D$=Helmholtz free energy, isotropic part, deviatoric part
$A, B^d, C^d, \underset{\approx}{D}, \underset{\approx}{E}^s$,
$\underset{\approx}{F}^s, D_2, E_2^s, \mu_s, \rho_s$,
$\underset{\approx}{k}^d, \lambda^d, \boldsymbol{b}^i, a^i, b^i$=constants of free energy function and evolution equations
$\boldsymbol{q}^n, \boldsymbol{p}^s, q_{kk}^d$=internal state variables
$Z_s, Z_D, \zeta_s, \zeta_D$=intrinsic times
$F(\zeta_s)$,
$f_h(Z_s), f_s(Z_s)$=hardening-softening function, hardening part softening part
$F(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \zeta_s)$,
$f(\eta), F_\eta(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}), \beta_1$,
$\beta_2, a, a_1, a_2, a_3$=hardening-softening function proposed by Bažant et al. and its parameters
β_s, β_h=parameters of hardening-softening function proposed by Wu et al.
Z_{scr}, ζ_{scr}=values of Z_s and ζ_s which divide softening response and hardening one
α_s, β_s=parameters of hardening-softening function
ξ=parameter related to densification
$\rho_1(Z), R_r, \beta_r$=kernel function of deviatoric equation and its parameters
$\alpha, \rho_0, \mu_1, \mu_2$=parameters associated with deviatoric response
R, c, ρ, λ=parameters associated with pore pressure response
$Q=c-R$
$\bar{s}_\mu$=undrained shear stress strength
$\bar{u}_f$=final value of undrained pore pressure strength
ε_{le}^p=plastic deviatoric strain making $d\bar{s}/d\varepsilon^p=0$
Z_{le}, ε_{le}=values of Z_s and ε_a corresponding to ε_{le}^p
m_1, m_2=material constants
f_1, f_2, f_3, f_4, f_5=material functions
$2G^p=(d\bar{s}/d\varepsilon^p)_{\varepsilon^p=0}$
$D_0=(d\bar{u}/d\varepsilon_a)_{\varepsilon_a=0}$
$D_f=(d\bar{u}/d\varepsilon_a)_{\varepsilon_a=\varepsilon_{le}}$
$Y=\exp[(\alpha-\beta_s)Z_s]$

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